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Pseudo-anti commuting Ricci tensor and Ricci soliton real hypersurfaces in the complex quadric ☆

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A B S T R A C T

First we introduce a new notion of pseudo-anti commuting Ricci tensor for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_2SO_m$ and give a complete classification of these hypersurfaces in the complex quadric Q^m. Next as an application we give a complete classification of Ricci solitons on Hopf real hypersurfaces in the complex quadric Q^m.

R É S U M É

Tout d'abord, on introduit notion de tenseur Ricci pseudo-anti commutatif pour les hypersurfaces réelles de la quadrique complexe $Q^m = SO_{m+2}/SO_mSO_2$, et on déduit une classification complète des ces hypersurfaces de la quadrique complexe Q^m. Ensuite, en tant qu'application, on en déduit une classification complète des solitons hypersurfaces réelles Ricci sur Hopf de la quadrique complexe Q^m.

1. Introduction

In the case of Hermitian symmetric spaces of rank 2, usually we can give examples of Riemannian symmetric spaces $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ and $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$, which are called complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [9–11,15–17]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$. The rank of this kind of Hermitian symmetric spaces is equal to 2.

Among the other different types of Hermitian symmetric spaces with rank 2 in the class of compact type manifolds, we can give the example of complex quadric $Q^m = SO_{m+2}/SO_2SO_m$. It is a complex hypersurface

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in complex projective space $\mathbb{C}P^m$ (see Smyth [14,19,20]). The complex quadric Q^m is considered as a real Grassmann manifold of compact type with rank 2 (see Kobayashi and Nomizu [5]). Moreover, it is well known that the complex quadric admits two important geometric structures, a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [4] and Reckziegel [12]).

Applying the Kähler structure J of the complex quadric Q^m, we can transfer any tangent vector fields X on M in Q^m as follows:

$$JX = \phi X + \eta(X)N,$$

where $\phi X = (JX)^T$ denotes the tangential component of JX, η an 1-form defined by $\eta(X) = g(JX, N) = g(X, \xi)$ for the Reeb vector field $\xi = -JN$ and N is a unit normal vector field on M in Q^m.

When the Ricci tensor Ric commutes or anti-commutes with the structure tensor ϕ, that is, $\mathrm{Ric}\cdot\phi = \phi\cdot\mathrm{Ric}$ or $\mathrm{Ric}\cdot\phi = -\phi\cdot\mathrm{Ric}$, the Ricci tensor is said to be commuting or anti-commuting respectively. Motivated by the notions of commuting and anti-commuting, we consider a new notion of *pseudo-anti commuting* Ricci tensor which was introduced in a paper due to Jeong and Suh [3]. It is defined by

$$\mathrm{Ric}\cdot\phi + \phi\cdot\mathrm{Ric} = \kappa\phi, \quad \kappa \neq 0 : \text{constant},$$

where the structure tensor ϕ is induced from the Kähler structure J of the Hermitian symmetric space.

If the Ricci tensor of a real hypersurface M in Q^m satisfies

$$\mathrm{Ric}(X) = aX + b\eta(X)\xi,$$

for any vector fields X on M and constants $a, b \in R$, then M is said to be *pseudo-Einstein*.

It is known that Einstein, or pseudo-Einstein real hypersurfaces in the sense of Besse [1], Kon [6], and Cecil and Ryan [2], satisfy the condition of pseudo-anti commuting Ricci tensor. A real hypersurface of type (B) in $\mathbb{C}P^m$, which is characterized by $S\phi + \phi S = k\phi$, $k \neq 0$, where S denotes the shape operator of M in $\mathbb{C}P^m$, and are tubes over a totally real totally geodesic real projective space $\mathbb{R}P^n$, $m = 2n$, satisfy the formula of pseudo-anti commuting Ricci tensor (see Yano and Kon [21]). Moreover, it can be easily checked that Einstein hypersurfaces and some special kind of pseudo-Einstein hypersurfaces in $G_2(\mathbb{C}^{m+2})$, and hypersurfaces of type (B) in $G_2(\mathbb{C}^{m+2})$, which are tubes over a totally real totally geodesic quaternionic projective space $\mathbb{H}P^n$, $m = 2n$, satisfy this formula (see Pérez, Suh and Watanabe [11], Suh [15] and [18]).

In the complex quadric Q^m, Suh [18,19] classified all *contact* hypersurfaces in Q^m, which satisfy $S\phi + \phi S = k\phi$, $k \neq 0$, for the shape operator S of a real hypersurface M in Q^m, and has given a characterization of a tube of radius r around an m-dimensional totally real and totally geodesic sphere S^m in Q^m. All of these hypersurfaces in Hermitian symmetric spaces also satisfy the condition of pseudo-anti commuting Ricci tensor.

Recently, we have shown that a solution of the Ricci flow equation $\frac{\partial}{\partial t}g(t) = -2\mathrm{Ric}(g(t))$ is given by

$$\frac{1}{2}(\mathfrak{L}_V g)(X, Y) + \mathrm{Ric}(X, Y) = \rho g(X, Y),$$

where ρ is a constant and $\mathfrak{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Morgan and Tian [7]). Then the solution is said to be a *Ricci soliton* with potential vector field V and Ricci soliton constant ρ, and surprisingly, it satisfies the pseudo-anti commuting condition $\mathrm{Ric}\cdot\phi + \phi\cdot\mathrm{Ric} = \kappa\phi$, where $\kappa = 2\rho$ is a non-zero constant.

In the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, Jeong and Suh [3] gave a classification of Ricci solitons for real hypersurfaces. In this paper we want to give a complete classification of pseudo-anti commuting

Ricci tensor for Hopf real hypersurfaces in the complex quadric Q^m. In order to do this we introduce some background for the study of real hypersurfaces in Hermitian symmetric spaces including complex projective space $\mathbb{C}P^m = SU_{m+1}/S(U_1U_m)$, complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$, complex hyperbolic two-plane Grassmannian $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$ and real two-plane Grassmannian, that is, a complex quadric Q^m.

Okumura [13] proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_1U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^m$ for some $k \in \{0, \dots, m-1\}$. For the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ a classification was obtained by Suh in [15] and [17]. The Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$. Moreover, in [16] we have proved that the Reeb flow on a real hypersurface in $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$. For the complex quadric $Q^m = SO_{m+2}/SO_2SO_m$, Suh [18] and [19] has obtained the following result:

Theorem A. *Let M be a real hypersurface of the complex quadric Q^m, $m \geq 3$. Then the Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

On the other hand, when a real hypersurface M in the complex quadric Q^m satisfies the formula $S\phi + \phi S = k\phi$, $k \neq 0$ constant, for the shape operator S, we say that M is a *contact* real hypersurface in Q^m. In the papers due to Suh [19] and [20], we have introduced the classification of contact real hypersurfaces in Q^m as follows:

Theorem B. *Let M be a connected orientable real hypersurface with constant mean curvature in the Hermitian symmetric space $Q^m = SO_{m+2}/SO_mSO_2$, $m \geq 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of a tube of radius r, $0 < r < \frac{\pi}{2\sqrt{2}}$, around the m-dimensional sphere S^m which is embedded in Q^m as a real form of Q^m.*

In addition to the complex structure J there is another distinguished geometric structure on Q^m, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^m. The set is denoted by $\mathfrak{A}_{[z]} = \{A_{\lambda\bar{z}} | \lambda \in S^1 \subset \mathbb{C}\}$, $[z] \in Q^m$, and it is the set of all complex conjugations defined on Q^m. Then $\mathfrak{A}_{[z]}$ becomes a parallel rank 2-subbundle of End TQ^m. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^m. Here the notion of parallel vector bundle $\mathfrak{A}$ means that $(\bar{\nabla}_X A)Y = q(X)AY$ for any vector fields X and Y on Q^m, where $\bar{\nabla}$ and q denote a connection and a certain 1-form defined on $T_{[z]}Q^m$, $[z] \in Q^m$ respectively.

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. Here maximal flat means a 2-dimensional curvature flat maximal totally geodesic submanifold in Q^m. Such a maximal flat always exists, because the rank of Q^m is 2. There are two types of singular tangent vectors for the complex quadric Q^m as follows:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

Here we note that the unit normal N is said to be $\mathfrak{A}$-principal if N is invariant under the complex conjugation $A \in \mathfrak{A}$, that is, $AN = N$.

Now at each point $z \in M$ let us consider a maximal $\mathfrak{A}$-invariant subspace

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_{[z]}\}$$

of T_zM, $z{\in}M$. Thus for a case where the unit normal vector field N is $\mathfrak{A}$-isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}_z^{\perp} = \mathcal{C}_z \ominus \mathcal{Q}_z$, $z{\in}M$, of the distribution $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$, becomes $\mathcal{Q}_z^{\perp} = \text{Span}\,[A\xi, AN]$. Here it can be easily checked that the vector fields $A\xi$ and AN belong to the tangent space T_zM, $z{\in}M$ if the unit normal vector field N becomes $\mathfrak{A}$-isotropic. Then in this paper we give a complete classification for real hypersurfaces with pseudo-anti commuting Ricci tensor in the complex quadric Q^m as follows:

Main Theorem 1. *Let M be a Hopf real hypersurface with pseudo-anti commuting Ricci tensor in the complex quadric Q^m, $m{\geq}3$. Then M is locally congruent to one of the following:*

(i) *M is an open part of a tube of radius r, $0 < r < \frac{\pi}{2\sqrt{2}}$, around a totally real and totally geodesic m-dimensional unit sphere S^m in Q^m, with $\mathfrak{A}$-principal unit normal vector field.*
(ii) *M is an open part of a tube of radius r, $0 < r < \frac{\pi}{2}$, $r{\neq}\frac{\pi}{4}$, around a totally geodesic k-dimensional complex projective space $\mathbb{C}P^k$ in Q^{2k}, $m = 2k$. Here the unit normal vector field N is $\mathfrak{A}$-isotropic.*

When we consider the Ricci soliton (M, g, ξ, ρ) on a real hypersurface in the complex quadric Q^m, it can be easily checked that (M, g, ξ, ρ) satisfies the condition of pseudo-anti commuting Ricci tensor, that is, $\text{Ric}{\cdot}\phi + \phi{\cdot}\text{Ric} = \kappa\phi$, $\kappa = 2\rho{\neq}0$ constant. So, naturally the classification results in Main Theorem 1 can be used to study a Ricci soliton (M, g, ξ, ρ). Then by virtue of Theorems A and B, and Main Theorem 1 we can state another theorem on Ricci solitons as follows:

Main Theorem 2. *Let (M, g, ξ, ρ) be a Ricci soliton on a Hopf real hypersurface in the complex quadric Q^m, $m{\geq}3$. Then M is locally congruent to one of the following:*

(i) *M is a tube of radius r around a totally real and totally geodesic m-dimensional unit sphere S^m in Q^m, with radius $r = \frac{1}{\sqrt{2}} \cot^{-1}\left(\frac{1}{2\sqrt{2}(m-1)}\right)$ or $r = \frac{1}{\sqrt{2}} \cot^{-1}\left(\frac{1}{2\sqrt{2}m}\right)$. Here the unit normal vector field N is $\mathfrak{A}$-principal.*
(ii) *M is a tube of radius $r = \tan^{-1}\sqrt{\frac{k}{k-1}}$ around a totally geodesic k-dimensional complex projective space $\mathbb{C}P^k$ in Q^{2k}, $m = 2k$. Here the unit normal vector field N is $\mathfrak{A}$-isotropic.*

2. The complex quadric

For more details not contained in this section we refer to [4,12,18–20]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where $z_1, \ldots, z_{m+2}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J, g) on the complex quadric. For each $[z] \in Q^m$ we identify $T_{[z]}\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus \mathbb{C}\bar{z}$ of $\mathbb{C}z$ in $\mathbb{C}^{m+2}$ (see Kobayashi and Nomizu [5]). The tangent space $T_{[z]}Q^m$ can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus (\mathbb{C}z \oplus \mathbb{C}\bar{z})$ of $\mathbb{C}z \oplus \mathbb{C}\bar{z}$ in $\mathbb{C}^{m+2}$, where $\bar{z} \in \nu_{[z]}Q^m$ is a normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point $[z]$.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2}, namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_1U_{m+1})$. We denote by $o = [0, \ldots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_1U_{m+1})$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$.

The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_2SO_m$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of Q^m as a Hermitian symmetric space of rank 2. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on.

We denote by $A_{\bar{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to the unit normal $\bar{z}$. It is defined by $A_{\bar{z}}w = \bar{\nabla}_w \bar{z} = \bar{w}$ for a complex Euclidean connection $\bar{\nabla}$ induced from $\mathbb{C}^{m+2}$ and all $w \in T_{[z]}Q^m$. That is, the shape operator $A_{\bar{z}}$ is just a complex conjugation restricted to $T_{[z]}Q^m$. Moreover, it satisfies the following for any $w \in T_{[z]}Q^m$ and any $\lambda \in S^1 \subset \mathbb{C}$

$$\begin{aligned} A^2_{\lambda\bar{z}}w &= A_{\lambda\bar{z}}A_{\lambda\bar{z}}w = A_{\lambda\bar{z}}\lambda\bar{w} \\ &= \lambda A_{\bar{z}}\lambda\bar{w} = \lambda\bar{\nabla}_{\lambda\bar{w}}\bar{z} = \lambda\bar{\lambda}\bar{\bar{w}} \\ &= |\lambda|^2 w = w. \end{aligned}$$

Accordingly, $A^2_{\lambda\bar{z}} = I$ for any $\lambda \in S^1$. So the shape operator $A_{\bar{z}}$ becomes an anti-commuting involution such that $A^2_{\bar{z}} = I$ and $A_{\bar{z}}J = -JA_{\bar{z}}$ on the complex vector space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}}) = i\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (-1)-eigenspace of $A_{\bar{z}}$. That is, $A_{\bar{z}}X = X$ and $A_{\bar{z}}JX = -JX$, respectively, for any $X \in V(A_{\bar{z}})$.

Geometrically this means that the shape operator $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^m$, or equivalently, is a complex conjugation on $T_{[z]}Q^m$. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces a *parallel* S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations.

There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m-dimensional sphere S^m. Through each point $[z] \in Q^m$ there exists a one-parameter family of real forms of Q^m which are isometric to the sphere S^m. These real forms are congruent to each other under action of the center SO_2 of the isotropy subgroup of SO_{m+2} at $[z]$. The isometric reflection of Q^m in such a real form S^m is an isometry, and the differential at $[z]$ of such a reflection is a conjugation on $T_{[z]}Q^m$. Thus the family $\mathfrak{A}$ of conjugations on $T_{[z]}Q^m$ corresponds to the family of real forms S^m of Q^m containing z, and the subspaces $V(A) \subset T_zQ^m$ correspond to the tangent spaces T_zS^m of the real forms S^m of Q^m.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned} \bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY. \end{aligned}$$

Note that J and each complex conjugation A anti-commute, that is, $AJ = -JA$ for each $A \in \mathfrak{A}$.

3. Some general equations

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. The tangent bundle TM of M

splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define a maximal $\mathfrak{A}$-invariant subspace of T_zM, $z{\in}M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Lemma 3.1. *([18]) For each $z \in M$ we have*

(i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*
(ii) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

We now assume that M is a Hopf hypersurface. Then the shape operator S of M in Q^m satisfies

$$S\xi = \alpha\xi,$$

where $\alpha = g(S\xi, \xi)$ denotes the Reeb function for the Reeb vector field $\xi = -JN$ on M.

When we consider a transform JX of the Kähler structure J on Q^m for any vector field X on M in Q^m, we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M. Then we consider the Codazzi equation

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) = \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) + g(X, AN)g(AY, Z) \\ - g(Y, AN)g(AX, Z) + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned}$$

Putting $Z = \xi$ we get

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = -2g(\phi X, Y) + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) &= g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

Reinserting this into the previous equation yields

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = -2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(\xi, AN)g(Y, A\xi)\eta(X) \\ - 2g(Y, AN)g(\xi, A\xi)\eta(X) + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Altogether this implies

$$
\begin{aligned}
0 = {} & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\
& - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) + 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\
& - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X).
\end{aligned}
$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see Proposition 3 in [12]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$
\begin{aligned}
N &= \cos(t)Z_1 + \sin(t)JZ_2, \\
AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\
\xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\
A\xi &= \sin(t)Z_2 + \cos(t)JZ_1.
\end{aligned}
$$

This implies $g(\xi, AN) = 0$ and hence

$$
\begin{aligned}
0 = {} & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\
& - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) - 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X).
\end{aligned}
$$

4. Key lemma

By the equation of Gauss, the curvature tensor $R(X,Y)Z$ for a real hypersurface M in Q^m induced from the curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$ as follows:

$$
\begin{aligned}
R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z + g(AY, Z)AX \\
& - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY + g(SY, Z)SX - g(SX, Z)SY
\end{aligned}
$$

for any $X, Y, Z \in T_zM$, $z \in M$.

From this, contracting Y and Z on M in Q^m, for a real hypersurface M in Q^m we have

$$
\begin{aligned}
\mathrm{Ric}(X) = {} & (2m-1)X - X - \phi^2 X - 2\phi^2 X - g(AN, N)AX - X + g(AX, N)AN - g(JAN, N)JAX \\
& - X + g(JAX, N)JAN + (\mathrm{Tr}\, S)SX - S^2 X \\
= {} & (2m-1)X - 3\eta(X)\xi - g(AN, N)AX + g(AX, N)AN - g(JAN, N)JAX + g(JAX, N)JAN \\
& + (\mathrm{Tr}\, S)SX - S^2 X
\end{aligned}
\tag{4.1}
$$

where $\mathrm{Tr}\, S$ denotes the trace of the shape operator S and we have used the following

$$
\begin{aligned}
\sum_{i=1}^{2m-1} g(Ae_i, e_i) &= \mathrm{Tr}\, A - g(AN, N) = -g(AN, N), \\
\sum_{i=1}^{2m-1} g(AX, e_i)Ae_i &= \sum_{i=1}^{2m} g(AX, e_i)Ae_i - g(AX, N)AN = X - g(AX, N)AN, \\
\sum_{i=1}^{2m-1} g(JAe_i, e_i)JAX &= \sum_{i=1}^{2m} g(JAe_i.e_i)JAX - g(JAN, N)JAX,
\end{aligned}
$$

and

$$\begin{aligned}\sum_{i=1}^{2m-1} g(JAX, e_i)JAe_i &= \sum_{i=1}^{2m} g(JAX, e_i)JAe_i - g(JAX, N)JAN \\ &= JAJAX - g(JAX, N)JAN \\ &= X - g(JAX, N)JAN.\end{aligned}$$

Now we want to check whether a pseudo-Einstein real hypersurface or a contact hypersurface in the complex quadric Q^m has pseudo-anti commuting Ricci tensor or not.

Example 4.1. Let M be a pseudo-Einstein real hypersurface in Q^m. The Ricci tensor is given by $\mathrm{Ric}(X) = aX + b\eta(X)\xi$. Then $\mathrm{Ric}(\phi X) = a\phi X$ and $\phi\mathrm{Ric}(X) = a\phi X$. This implies $\mathrm{Ric}\cdot\phi + \phi\cdot\mathrm{Ric} = \kappa\phi$, $\kappa = 2a$. So M satisfies the pseudo-anti commuting Ricci tensor property.

Example 4.2. When we consider a contact hypersurface M in the complex quadric Q^m, M is locally congruent to a tube of radius r, $0 < r < \frac{\pi}{2\sqrt{2}}$, over a totally geodesic and totally real space form S^m in Q^m (see Suh [18] and [19]). In [18] and [19] it is also shown that M has three distinct constant principal curvatures $\alpha = -\sqrt{2}\cot(\sqrt{2}r)$, $\lambda = 0$ and $\mu = \sqrt{2}\tan(\sqrt{2}r)$ with multiplicities 1, $m-1$ and $m-1$ respectively. This is equivalent to $\phi S + S\phi = k\phi$, where $k \neq 0$ is a constant. Moreover, the unit normal N of M in Q^m is $\mathfrak{A}$-principal, that is, $AN = N$, and $A\xi = -\xi$. Then the Ricci tensor becomes

$$\mathrm{Ric}(X) = (2m-1)X - 2\eta(X)\xi - AX + hSX - S^2X$$

where $h = \mathrm{Tr}\, S$ is defined as the trace of the shape operator S on M and denotes the mean curvature of M in Q^m.

From this it follows that

$$\begin{aligned}(\mathrm{Ric}\cdot\phi + \phi\cdot\mathrm{Ric})X = (4m-2)\phi X &- (A\phi + \phi A)X \\ &+ h(S\phi + \phi S)X - (S^2\phi + \phi S^2)X.\end{aligned}$$

Since $S\phi + \phi S = k\phi$ implies $S\phi S + \phi S^2 = k\phi S$ and $S^2\phi + S\phi S = kS\phi$ respectively, we get the following:

$$S^2\phi + 2S\phi S + \phi S^2 = k(\phi S + S\phi).$$

On the other hand, in Suh [18] and [19] we saw the following for contact hypersurfaces in Q^m with $\mathfrak{A}$-principal normal vector field

$$\begin{aligned}2S\phi S &= \alpha(S\phi + \phi S)X + 2\phi X \\ &= (\alpha k + 2)\phi X \\ &= 0,\end{aligned}$$

where we have used $\alpha k = -2$. Then by the property $(A\phi + \phi A)X = 0$ for any vector field X on M in Q^m, it follows that

$$(\mathrm{Ric}\cdot\phi + \phi\cdot\mathrm{Ric})X = \{(4m-2) + hk - k^2\}\phi X.$$

Here from the anti-commutativity $AJ = -JA$ between the Kähler structure J and the complex conjugation A we note that for the $\mathfrak{A}$-principal unit normal vector field

$$\begin{aligned}0 &= AJX + JAX \\ &= A(\phi X + \eta(X)N) + \phi AX + \eta(AX)N \\ &= A\phi X + \phi AX + \eta(X)N + \eta(AX)N.\end{aligned}$$

It follows that $(A\phi + \phi A)X = 0$, because $g(A\phi X, N) = g(\phi X, N) = 0$ and $g(AX, N) = g(X, AN) = g(X, N) = 0$ for any tangent vector field X on M.

Now we give an important proposition which will be used in the proof of our Main Theorem 2 as follows:

Proposition 4.3. *Let M be a Hopf real hypersurface with pseudo-anti commuting Ricci tensor in the complex quadric Q^m. Then the unit normal N becomes singular, that is, N is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Proof. By putting $X = \xi$ in (4.1), we have the following

$$\begin{aligned}\mathrm{Ric}(\xi) = (2m-1)\xi - 3\xi - g(AN, N)A\xi - g(JAN, N)JA\xi \\ + g(JA\xi, N)JAN + hS\xi - S^2\xi.\end{aligned}$$

Now let us put $X = \xi$ into the condition of pseudo-anti commutativity $\mathrm{Ric}\cdot\phi + \phi\cdot\mathrm{Ric} = \kappa\phi$. We have

$$0 = \phi\cdot\mathrm{Ric}(\xi) = -g(AN, N)\phi A\xi + g(JA\xi, N)\phi JAN = -2g(AN, N)\phi A\xi.$$

This gives $g(AN, N) = 0$ or $A\xi = \eta(A\xi)\xi$. From the first case we know that the unit normal vector field N is $\mathfrak{A}$-isotropic. In the second case, the involution property of the complex conjugation A on Q^m gives $\xi = A^2\xi = \beta A\xi = \beta^2\xi$, where we have put $\beta = g(A\xi, \xi)$. This gives $\beta = \pm 1$. Now let us consider $\beta = -1$. Then $A\xi = -\xi = JN$ and $A\xi = -AJN = JAN$. This means $AN = N$, that is, the unit normal vector field N is $\mathfrak{A}$-principal. □

By virtue of this proposition, naturally we consider two cases, that N is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal for real hypersurfaces with pseudo-anti commuting Ricci tensor in Q^m. So in section 5 we give a complete classification of pseudo-anti commuting real hypersurfaces in Q^m with $\mathfrak{A}$-principal unit normal vector field, and in section 6 we will complete our Main Theorem 2 for the case of $\mathfrak{A}$-isotropic unit normal vector field.

In the proof of our Main Theorems 1 and 2, we want to give more information on Hopf hypersurfaces in the complex quadric with $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic normal vector field. Using the formulas given in section 3 we can prove two important lemmas as follows:

Lemma 4.4. *([18]) Let M be a Hopf hypersurface in Q^m such that the normal vector field N is $\mathfrak{A}$-principal everywhere. Then α is constant. Moreover, if $X \in \mathcal{C}$ is a principal vector field of M with principal curvature λ, then $2\lambda \neq \alpha$ and ϕX is a principal vector field of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

Lemma 4.5. *([18]) Let M be a Hopf hypersurface in Q^m, $m \geq 3$, such that the normal vector field N is $\mathfrak{A}$-isotropic everywhere. Then α is constant.*

5. Pseudo-anti commuting Ricci tensor for real hypersurfaces with $\mathfrak{A}$-principal normal vector field

In this section we consider an $\mathfrak{A}$-principal normal vector field N, that is, $AN = N$, for a real hypersurface M in Q^m. Then (4.1) becomes

$$\mathrm{Ric}(X) = (2m-1)X - 2\eta(X)\xi - AX + hSX - S^2X \tag{5.1}$$

where $h = \operatorname{Tr} S$ denotes the mean curvature of M in Q^m, defined as the trace of the shape operator S on M. Then from this, by differentiating the Ricci tensor, we have

$$\begin{aligned}(\nabla_X \mathrm{Ric})Y &= -2g(\nabla_X \xi, Y)\xi - 2\eta(Y)\nabla_X \xi - (\nabla_X A)Y + (Xh)SY + h(\nabla_X S)Y - (\nabla_X S^2)Y \\ &= -2g(\phi SX, Y)\xi - 2\eta(Y)\phi SX - (\nabla_X A)Y + (Xh)SY + h(\nabla_X S)Y - (\nabla_X S^2)Y,\end{aligned} \tag{5.2}$$

where $(\nabla_X A)Y = \nabla_X(AY) - A\nabla_X Y$. Here, AY belongs to T_zM, $z \in M$, from the fact that $g(AY, N) = g(Y, AN) = g(Y, N) = 0$ for any tangent vector Y on M.

Now differentiate the condition of pseudo-anti commuting Ricci tensor as follows:

$$(\nabla_X \mathrm{Ric})\phi Y + \mathrm{Ric}((\nabla_X \phi)Y) + (\nabla_X \phi)(\mathrm{Ric}(Y)) + \phi(\nabla_X \mathrm{Ric})Y = k(\nabla_X \phi)Y.$$

Then the first term becomes

$$\begin{aligned}(\nabla_X \mathrm{Ric})\phi Y = -2g(\phi SX, Y)\xi - 2\eta(\phi Y)\nabla_X \xi - (\nabla_X A)\phi Y + (Xh)S\phi Y \\ + h(\nabla_X S)\phi Y - (\nabla_X S^2)\phi Y.\end{aligned} \tag{5.3}$$

The second term is

$$\begin{aligned}\mathrm{Ric}((\nabla_X \phi)Y) = \eta(Y)\big\{(2m-1)SX - 2\alpha\eta(X)\xi - ASX + hS^2X - S^3X\big\} \\ - g(SX, Y)\{2(m-1)\xi + (h\alpha - \alpha^2)\xi\}.\end{aligned} \tag{5.4}$$

The third term becomes

$$\begin{aligned}(\nabla_X \phi)(\mathrm{Ric}(Y)) &= \eta(\mathrm{Ric}(Y))SX - g(SX, \mathrm{Ric}(Y))\xi \\ &= \big\{2(m-1) + h\alpha - \alpha^2\big\}\eta(Y)SX \\ &\quad -\big\{(2m-1)g(SX, Y) - 2\alpha\eta(Y)\eta(X) - g(SX, AY) + hg(SX, SY) - g(SX, S^2Y)\big\}.\end{aligned} \tag{5.5}$$

Finally the fourth term is given by

$$\phi(\nabla_X \mathrm{Ric})Y = -2\eta(Y)\phi^2 SX - \phi(\nabla_X A)Y + (Xh)\phi SY + h\phi(\nabla_X S)Y - \phi(\nabla_X S^2)Y. \tag{5.6}$$

Summing up all the above terms, we have the following:

$$\begin{aligned}&- 2g(\phi SX, Y)\xi - (\nabla_X A)\phi Y + (Xh)S\phi Y + h(\nabla_X S)\phi Y - (\nabla_X S^2)\phi Y \\ &\quad + \eta(Y)\big\{(2m-1)SX - 2\alpha\eta(X)\xi - ASX + hS^2X - S^3X\big\} \\ &\quad - g(SX, Y)\{2(m-1)\xi + (h\alpha - \alpha^2)\xi\} \\ &\quad + \big\{2(m-1) + h\alpha - \alpha^2\big\}\eta(Y)SX \\ &\quad - \big\{(2m-1)g(SX, Y) - 2\alpha\eta(Y)\eta(X) - g(SX, AY) \\ &\quad + hg(SX, SY) - g(SX, S^2Y)\big\}\xi \\ &\quad - 2\eta(Y)\phi^2 SX - \phi(\nabla_X A)Y + (Xh)\phi SY \\ &\quad + h\phi(\nabla_X S)Y - \phi(\nabla_X S^2)Y \\ &= \kappa\big\{\eta(Y)SX - g(SX, Y)\xi\big\}.\end{aligned} \tag{5.7}$$

Moreover, we get the following from the assumption of Hopf

$$(\nabla_X S)\xi = \nabla_X(S\xi) - S\nabla_X\xi = (X\alpha)\xi + \alpha\phi SX - S\phi SX,$$

and

$$(\nabla_X S^2)\xi = \nabla_X(S^2\xi) - S^2\nabla_X\xi = (X\alpha^2)\xi + \alpha^2\phi SX - S^2\phi SX.$$

Then it follows that

$$g((\nabla_X S)\phi Y, \xi) = g(\phi Y, (\nabla_X S)\xi) = \alpha g(\phi SX, \phi Y) - g(S\phi SX, \phi Y)$$

and

$$g((\nabla_X S^2)\phi Y, \xi) = g(\phi Y, (\nabla_X S^2)\xi) = \alpha^2 g(\phi SX, \phi Y) - g(S^2\phi SX, \phi Y).$$

The inner product of (5.7) with the Reeb vector field ξ while using the above formulas yields

$$\begin{aligned}
&-2g(\phi SX, Y) + h\alpha g(\phi SX, \phi Y) - hg(S\phi SX, \phi Y)\\
&\quad -\alpha^2 g(\phi SX, \phi Y) + g(S^2\phi SX, \phi Y)\\
&\quad +\eta(Y)\{2(m-1)\alpha\eta(X) + (h\alpha^2 - \alpha^3)\eta(X)\}\\
&\quad -g(SX, Y)\{2(m-1) + (h\alpha - \alpha^2)\}\\
&\quad +\{2(m-1) + (h\alpha - \alpha^2)\}\alpha\eta(X)\eta(Y)\\
&\quad -\{(2m-1)g(SX, Y) - 2\alpha\eta(Y)\eta(X) - g(SX, AY)\\
&\quad +hg(SX, SY) - g(SX, S^2Y)\}\xi\\
&= \kappa\{\alpha\eta(X)\eta(Y) - g(SX, Y)\}.
\end{aligned} \tag{5.8}$$

Now let us put $SX = \lambda X$, $X \in T_\lambda$, where X is orthogonal to the Reeb vector field ξ, and $S\phi X = \mu\phi X$, and $Y = X$ in (5.8). Then by Lemma 4.4 in section 4, we have

$$\Big[-2 + h\alpha - h\mu - \alpha^2 + \mu^2 - \{2(m-1) + h\alpha - \alpha^2\} - \{(2m-1) - g(X, AX) + h\lambda - \lambda^2\}\Big]\lambda = -\kappa\lambda.$$

For a non-vanishing principal curvature λ, it can be rewritten as follows:

$$\lambda^2 + \mu^2 - h(\lambda + \mu) - \{4m - 1 - g(X, AX)\} + \kappa = 0. \tag{5.9}$$

In order to prove our Main Theorem 1, we consider the condition of pseudo-anti commuting Ricci tensor. Then it follows that

$$\begin{aligned}
\mathrm{Ric}(\phi X) + \phi\mathrm{Ric}(X) &= 2(2m-1)\phi X + h(S\phi + \phi S)X - (S^2\phi + \phi S^2)X\\
&= \{\kappa - 2(2m-1)\}\phi X.
\end{aligned} \tag{5.10}$$

From this, if we consider $X \in T_\lambda$ such that $SX = \lambda X$, $S\phi X = \mu\phi X$, $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$ in (5.10), we have

$$\lambda^2 + \mu^2 - h(\lambda + \mu) - \{2(2m-1) - \kappa\} = 0. \tag{5.11}$$

Comparing (5.9) and (5.11) for $\lambda \neq 0$, we have that for any $X \in T_\lambda$

$$g(X, AX) = 1. \tag{5.12}$$

This means that the eigenvector X in the principal curvature space T_λ belongs to the eigenspace $V(A)$ with complex conjugation A, that is, $X \in V(A)$, $AX = X$. Similarly, for the eigenvector $Y \in T_\mu$ with non-vanishing principal curvature $\mu \neq 0$ we have

$$\lambda^2 + \mu^2 - h(\lambda + \mu) - \{4m - 1 - g(Y, AY)\} + \kappa = 0. \tag{5.13}$$

From this, if we compare with (5.11), we know that for any eigenvector $Y \in T_\mu$, $\mu \neq 0$,

$$g(Y, AY) = 1. \tag{5.14}$$

Then (5.14) implies that the eigenvector $Y \in T_\mu$ belongs to the eigenspace $V(A)$, that is, $Y \in V(A)$, $AY = Y$. But the vector $Y \in T_\mu$ becomes $Y = \phi X$ for an eigenvector $X \in T_\lambda$. Then this gives $AY = A\phi X = -\phi AX = -\phi X = -Y$, that is, $Y \in JV(A)$, which gives a contradiction. Accordingly, we deduce that one of the principal curvatures vanishes. So let us say $\lambda = 0$. Then $\mu = -\frac{2}{\alpha}$. By Lemma 3.1, the expression of the shape operator S of M in Q^m becomes

$$S = \begin{bmatrix} \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -\frac{2}{\alpha} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & -\frac{2}{\alpha} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

This means that the shape operator satisfies $S\phi + \phi S = k\phi$, where $k = -\frac{2}{\alpha}$. Then by a theorem due to Suh [18] and [19], M is a tube of radius r around a totally geodesic and totally real m-dimensional sphere S^m in Q^m.

6. Pseudo-anti commuting Ricci tensor for real hypersurfaces with $\mathfrak{A}$-isotropic normal vector field

In this section we want to prove our Main Theorem 2 for real hypersurfaces with pseudo-anti commuting Ricci tensor in Q^m with $\mathfrak{A}$-isotropic unit normal vector field.

Before proving our Main Theorem 2 we prove a proposition

Proposition 6.1. *Let M be a Hopf real hypersurface with pseudo-anti commuting Ricci tensor in complex quadric Q^m, $m \geq 3$ with $\mathfrak{A}$-isotropic unit normal vector field. Then the distributions $\mathcal{Q}$ and $\mathcal{Q}^\perp = \mathcal{C} \ominus \mathcal{Q}$ are invariant under the shape operator S of M in Q^m.*

Proof. Since M is $\mathfrak{A}$-isotropic, by the formulas in section 3 we know that $g(A\xi, \xi) = 0$, $g(AN, N) = 0$ and $g(A\xi, N) = 0$. In this case the Ricci tensor becomes

$$\mathrm{Ric}(X) = (2m - 1)X - 3\eta(X)\xi + g(AX, N)AN + g(AX, \xi)A\xi + hSX - S^2X. \tag{6.1}$$

From this, the condition of pseudo-anti commuting Ricci tensor $\phi \cdot \mathrm{Ric}(X) + \mathrm{Ric}(\phi X) = \kappa\phi X$ is given by

$$\begin{aligned} \phi \cdot \mathrm{Ric}(X) + \mathrm{Ric}(\phi X) &= 2(2m - 1)\phi X - 2g(A\xi, X)AN + 2g(AN, X)A\xi \\ &\quad + h(\phi S + S\phi)X - (\phi S^2 + S^2\phi)X \\ &= \kappa\phi X \end{aligned} \tag{6.2}$$

for any X tangent to M. Substituting the vector fields $A\xi$ and AN respectively and using $\phi AN = A\xi$ and $\phi A\xi = -AN$, we have

$$\begin{aligned}\phi{\cdot}\mathrm{Ric}(A\xi) &= \mathrm{Ric}(AN) - \kappa AN \\ &= -2mAN + h\beta\phi A\xi - \beta^2\phi A\xi \\ &= \{-2m - h\beta + \beta^2\}AN,\end{aligned} \tag{6.3}$$

where the function β denotes $g(A\xi, \xi)$. This gives the following for some scalar functions δ and ν as follows:

$$\mathrm{Ric}(AN) = \delta AN \quad \text{and} \quad \mathrm{Ric}(A\xi) = \nu A\xi. \tag{6.4}$$

Now we consider a new symmetric operator T which is given by $T = hS - S^2$. Then by (6.4) we know that the new operator T preserves the distribution $Q^{\perp} = \mathrm{Span}[A\xi, AN]$, that is, $g(TQ, Q^{\perp}) = 0$. Then the commutativity, $ST = TS$, between the symmetric operator T and the shape operator S implies the existence of a common basis on M which simultaneously diagonalizes both operators. By virtue of this property we also have $g(SQ, Q^{\perp}) = 0$. This means that the distributions Q and $Q^{\perp}$ are invariant under the shape operator S of M in Q^m. This gives a complete proof of our Proposition. □

Since $g(AN, N) = 0$ for an $\mathfrak{A}$-isotropic normal vector field, we know that $AN = BN$ (see [18] and [19]), where BN denotes the tangential part of AN. It follows that

$$\nabla_Y(BN) = \nabla_Y(AN) = \{(\bar\nabla_Y A)N + A\bar\nabla_Y N\}^T = \{q(Y)JAN - ASY\}^T,$$

and

$$\begin{aligned}\nabla_Y(A\xi) &= \{(\bar\nabla_Y A)\xi + A\bar\nabla_Y\xi\}^T \\ &= \{q(Y)JA\xi + A\phi SY\}^T,\end{aligned}$$

where q is a certain 1-form defined on T_zM, $z{\in}M$ and $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^m. We take the derivative of the Ricci tensor as follows:

$$\begin{aligned}(\nabla_Y\mathrm{Ric})X &= \nabla_Y(\mathrm{Ric}(X)) - \mathrm{Ric}(\nabla_Y X) \\ &= -3(\nabla_Y\eta)(X)\xi - 3\eta(X)\nabla_Y\xi \\ &\quad + g(X, \nabla_Y(AN))AN + g(AX, N)\nabla_Y(AN) \\ &\quad + g((\nabla_Y(A\xi), X)A\xi + \eta(AX)\nabla_Y(A\xi) + (Yh)SX \\ &\quad + h(\nabla_Y S)X - (\nabla_Y S^2)X \\ &= -3g(\phi SY, X)\xi - 3\eta(X)\phi SY \\ &\quad + \{q(Y)g(JAN, X) - g(ASY, X)\}AN \\ &\quad + g(AX, N)\{q(Y)JAN - ASY\}^T + \{q(Y)g(JA\xi, X) \\ &\quad + g(A\phi SY, X)\}A\xi + \eta(AX)\{q(Y)JA\xi + A\phi SY\}^T \\ &\quad + (Yh)SX + h(\nabla_Y S)X - (\nabla_Y S^2)X.\end{aligned} \tag{6.5}$$

Using this formula, we will consider the derivative formula of the pseudo-anti commuting Ricci tensor property as follows:

$$(\nabla_Y \mathrm{Ric})\phi X + \mathrm{Ric}((\nabla_Y \phi)X) + (\nabla_Y \phi)\mathrm{Ric}(X) + \phi(\nabla_Y \mathrm{Ric})X = \kappa(\nabla_Y \phi)X.$$

Putting $X = \xi$ and using that the function α is constant in Lemma 4.5 in section 4 for $\mathfrak{A}$-isotropic unit normal vector field, it follows that

$$\begin{aligned}\{(2m-1)SY &- 3\alpha\eta(Y)\xi + g(ASY, N)AN + g(ASY,\xi)A\xi + hS^2Y - S^3Y\}\\ &- 2\alpha\eta(Y)\{2(m-2)\xi + (h\alpha - \alpha^2)\xi\} + \{2(m-2) + h\alpha - \alpha^2\}SY\\ &- 3\phi^2SY - g(ASY,\xi)\phi AN + g(A\phi SY,\xi)\phi A\xi\\ &- h\{\alpha\phi^2SY - \phi S\phi SY\} - \{\alpha^2\phi^2SY - \phi S^2\phi SY\}\\ &= \kappa\{SY - \alpha\eta(Y)\xi\}.\end{aligned} \tag{6.6}$$

By virtue of Proposition 6.1, we may put

$$SA\xi = \beta A\xi \quad \text{and} \quad SAN = \gamma AN.$$

Then substituting $SA\xi = \beta A\xi$ and $SAN = \gamma AN$ into (6.6), it follows that

$$\begin{aligned}\{(2m-1)SY &- 3\alpha\eta(Y)\xi + hS^2Y - S^3Y\}\\ &- 2\alpha\eta(Y)\{2(m-2)\xi + (h\alpha - \alpha^2)\xi\}\\ &+ \{2(m-2) + h\alpha - \alpha^2\}SY - 3\phi^2SY\\ &- h\{\alpha\phi^2SY - \phi S\phi SY\} - \{\alpha^2\phi^2SY - \phi S^2\phi SY\}\\ &= \kappa\{SY - \alpha\eta(Y)\xi\}.\end{aligned} \tag{6.7}$$

From this, by putting $Y = A\xi$ into (6.7), and using $\eta(SA\xi) = 0$ and $SA\xi = \beta A\xi$, we have

$$\begin{aligned}\{(2m-1)\beta A\xi &+ (h\beta^2 - \beta^3)A\xi\} + \{2(m-2) + h\alpha - \alpha^2\}\beta A\xi\\ &+ h\{\alpha\beta A\xi + \phi S\phi SA\xi\} + 3\beta A\xi + \{\alpha^2\beta A\xi + \phi S^2\phi SA\xi\}\\ &= \kappa\beta A\xi.\end{aligned} \tag{6.8}$$

We use the following formulas:

$$\phi S\phi SA\xi = -\beta\phi SAN = -\beta\gamma A\xi,$$

and

$$\phi S^2\phi SA\xi = -\beta\phi S^2AN = -\beta\gamma^2 A\xi,$$

because $A\xi = \phi AN$ and $\phi A\xi = -AN$. Then (6.8) becomes

$$\beta = 0 \tag{6.9}$$

or

$$\kappa = 4m - 2 + h\beta - \beta^2 + h\alpha - \alpha^2 + h(\alpha - \gamma) + (\alpha^2 - \gamma^2). \tag{6.10}$$

From these formulas we can prove the following lemma:

Lemma 6.2. *Let M be a Hopf real hypersurface with pseudo-anti commuting Ricci tensor in the complex quadric Q^m. If $SA\xi = \beta A\xi$ and $SAN = \gamma AN$, then we have the following:*

(i) *the mean curvature h is non-vanishing,*
(ii) *$\beta = \gamma = 0$ or $\alpha = \beta = \gamma$.*

Proof. We apply the pseudo-anti commuting Ricci tensor condition to the vector field $A\xi$ and use $\phi AN = A\xi$ and $\phi A\xi = -AN$. Then we have

$$\phi{\cdot}\mathrm{Ric}(A\xi) = \mathrm{Ric}(AN) - \kappa AN.$$

The left side of the above equation becomes

$$\begin{aligned}\phi{\cdot}\mathrm{Ric}(A\xi) &= -2mAN + h\beta\phi A\xi - \beta^2\phi A\xi \\ &= \{-2m - h\beta + \beta^2\}AN\end{aligned}$$

and the right side becomes

$$\mathrm{Ric}(AN) - \kappa AN = \{2m + h\gamma - \gamma^2\}AN - \kappa AN.$$

Consequently

$$4m + h(\beta + \gamma) - (\beta^2 + \gamma^2) = \kappa.$$

From this, compared with the formula (6.10), we deduce

$$h\alpha = h\gamma + 1.$$

Therefore $h\neq 0$. Similarly, if we apply the pseudo-anti commuting condition to AN, we find another formula

$$h\alpha = h\beta + 1.$$

From these two formulas, we infer that $h(\beta - \gamma) = 0$. From this, the mean curvature h is non-vanishing and $\beta = \gamma$.

Since the unit normal N is $\mathfrak{A}$-isotropic, we know that $g(\xi, A\xi) = 0$. Moreover, by Lemma 4.2 of [18], we have the following:

$$2S\phi SX = \alpha(S\phi + \phi S)X + 2\phi X - 2g(X, AN)A\xi + 2g(X, A\xi)AN. \tag{6.11}$$

Now let us consider the distribution $\mathcal{Q}^\perp$, which is an orthogonal complement of the maximal $\mathfrak{A}$-invariant subspace $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$ of T_zM, $z \in M$ in Q^m. Then by Lemma 3.1 in section 3, the orthogonal complement $\mathcal{Q}^\perp = \mathcal{C}\ominus\mathcal{Q}$ becomes $\mathcal{C}\ominus\mathcal{Q} = \mathrm{Span}\ [AN, A\xi]$. Then by Proposition 6.1, the distribution $\mathcal{Q}^\perp$ is invariant under the shape operator S. Then (6.11) gives the following for $SAN = \gamma AN$

$$\begin{aligned}(2\gamma - \alpha)S\phi AN &= (\alpha\gamma + 2)\phi AN - 2A\xi \\ &= (\alpha\gamma + 2)\phi AN - 2\phi AN \\ &= \alpha\gamma\phi AN.\end{aligned}$$

Here if $2\gamma - \alpha = 0$, then $\alpha\gamma = 2\gamma^2 = 0$. This means $\alpha = \gamma = 0$, which is in a contradiction to the above formula $h\alpha = h\gamma + 1$. From this, together with $A\xi = \phi AN$, we have the following

$$SA\xi = \frac{\alpha\gamma}{2\gamma - \alpha} A\xi.$$

From this we know that $\gamma = \beta$ and $\beta = \frac{\alpha\gamma}{2\gamma-\alpha}$ imply

$$\beta = \gamma = 0 \quad \text{or} \quad \gamma = \beta = \alpha. \tag{6.12}$$

This gives a complete proof of our Lemma. □

Now we assume $SY = \lambda Y$, $Y \in \mathcal{Q}$. Then (6.11) gives

$$S\phi Y = \mu\phi Y, \qquad \mu = \frac{\alpha\lambda + 2}{2\lambda - \alpha}.$$

In fact, (6.11) yields $(2\lambda - \alpha)S\phi Y = (\alpha\lambda + 2)\phi Y$ for any $Y \in \mathcal{Q}$, where Y is orthogonal to the vector fields AN and $A\xi$. Here, $2\lambda - \alpha$ is non-vanishing. Because, if $2\lambda - \alpha = 0$, then $\alpha\lambda + 2 = 2\lambda^2 + 2 = 0$, with contradiction.

Substituting these formulas into (6.6), we have the following:

$$\{(2m-1)\lambda + h\lambda^2 - \lambda^3\} + \{2(m-2) + h\alpha - \alpha^2\}\lambda + 3\lambda + h\alpha\lambda - h\lambda\mu + \alpha^2\lambda - \lambda\mu^2 = \kappa\lambda.$$

Without loss of generality, we can assume that one of the principal curvatures λ and μ is non-vanishing, because we can take $\mu = -\frac{2}{\alpha}$ if $\lambda = 0$. So let us say $\lambda \neq 0$. Then let us compare with the formulas from the derivative and the condition of pseudo-anti commuting Ricci tensor respectively as follows:

$$\begin{aligned} \kappa &= 2(2m-1) + h\lambda - \lambda^2 + h\alpha - \alpha^2 + h\alpha - h\mu + \alpha^2 - \mu^2 \\ &= 2(2m-1) + h(\lambda + \mu) - (\lambda^2 + \mu^2). \end{aligned}$$

This gives $h\alpha - h\mu = 0$. Similarly, for $SY = \mu Y$ we have $h\alpha - h\lambda = 0$. Then these two formulas give

$$h(\lambda - \mu) = 0.$$

Since we have noted that the mean curvature h is non-vanishing in Lemma 6.2, $\lambda = \mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. This implies that $\lambda^2 - \alpha\lambda - 1 = 0$. Accordingly $\lambda = \cot r$ or $-\tan r$. From this, together with Lemma 6.2, the shape operator S of Hopf hypersurface with pseudo-anti commuting Ricci tensor in Q^m can be expressed in the two cases as

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cot r & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cot r & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\tan r & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -\tan r \end{bmatrix},$$

or

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \alpha & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cot r & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cot r & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\tan r & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -\tan r \end{bmatrix}.$$

The first case means that the shape operator S and the structure tensor ϕ commute with each other, that is, $S\phi = \phi S$, which is equivalent to isometric Reeb flow on M in Q^m. Then by a result due to Suh [18] and [19], M is locally congruent to a tube of radius r around a totally geodesic complex projective space $P_k(\mathbb{C})$ in Q^{2k}. The second case also has the same property $S\phi = \phi S$. Then also by Suh [18] and [19], we know that $\alpha(= \beta = \gamma) = 2\cot 2r = 0$. That means $r = \frac{\pi}{4}$. In this case M is locally congruent to a tube of radius $r = \frac{\pi}{4}$ over a totally geodesic complex projective space $P_k(\mathbb{C})$ in Q^{2k}. That is, M is minimal. So by Lemma 6.2, this case does not occur. This completes the proof of our Main Theorem 1 for $\mathfrak{A}$-isotropic unit normal N.

7. Ricci solitons and pseudo-anti commuting Ricci tensor

In this section we want to introduce the notion of Ricci flow $\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}(g(t))$ and its solution named Ricci soliton (M, g, V, ρ) due to Morgan and Tian [7]. It was used to solve the Poincaré Conjecture by Perelman [8]. Next we will show that the Ricci soliton (M, g, V, ρ) satisfies the condition of pseudo-anti commuting Ricci tensor.

Now let us denote by (M, g) an m-dimensional Riemannian manifold. Then (M, g) is said to be a *Ricci soliton* if there exists a differentiable vector field V such that

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + \mathrm{Ric}(X, Y) = \rho g(X, Y), \tag{7.1}$$

for any vector fields $X, Y \in T_zM$, $z \in M^m$ and a constant ρ. In this case we say that (M, g, V, ρ) is a Ricci soliton with potential vector field V and Ricci soliton constant ρ. Depending on the Ricci soliton constant $\rho = 0$, $\rho < 0$ or $\rho > 0$, we say that the Ricci soliton (M, g, V, ρ) is *stable, expanding* or *shrinking*.

Now we assume that the potential vector field V coincides with the Reeb vector field ξ. Then (7.1) is equivalent to

$$\mathrm{Ric}(X, Y) + \frac{1}{2}g((\phi S - S\phi)X, Y) = \rho g(X, Y).$$

From this, $\mathrm{Ric}(X) = \frac{1}{2}(S\phi - \phi S)X + \rho X$. Then it gives respectively $\mathrm{Ric}(\phi X) = \rho\phi X + \frac{1}{2}(S\phi - \phi S)\phi X$ and $\phi\mathrm{Ric}(X) = \frac{1}{2}(\phi S\phi - \phi^2 S)X + \rho\phi X$. By the assumption of M being Hopf, we have the following

$$\begin{aligned} \mathrm{Ric}\cdot\phi(X) + \phi\cdot\mathrm{Ric}(X) &= 2\rho\phi X + \frac{1}{2}(S\phi^2 X - \phi^2 SX) \\ &= 2\rho\phi X + \frac{1}{2}\{S(-X + \eta(X)\xi) + SX - \eta(SX)\xi\} \\ &= 2\rho\phi X. \end{aligned} \tag{7.2}$$

So the Ricci soliton (M, g, ξ, ρ) satisfies the condition of pseudo-anti commuting Ricci tensor.

$$\mathrm{Ric}\cdot\phi + \phi\cdot\mathrm{Ric} = \kappa\phi, \quad \kappa = 2\rho \neq 0 : \text{constant}.$$

Now let us consider a Hopf real hypersurface M in the complex quadric Q^m, $m \geq 3$. Then the Hopf Ricci soliton (M, g, ξ, κ) satisfies the pseudo-anti commuting Ricci tensor property with Reeb potential vector field ξ. So by Proposition 4.3, the unit normal $N = J\xi$ to M in the complex quadric Q^m is singular, that is, N becomes $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. Then we assert the following:

Lemma 7.1. *Let M be a Hopf Ricci soliton real hypersurface in Q^m with potential Reeb field ξ. Then the Ricci soliton constant ρ is given by*

(i) *if N is $\mathfrak{A}$-principal*

$$\rho = 2(m-1) + h\alpha - \alpha^2,$$

(ii) *and if N is $\mathfrak{A}$-isotropic*

$$\rho = 2(m-2) + h\alpha - \alpha^2.$$

Proof. When the unit normal N is $\mathfrak{A}$-principal, the Ricci tensor becomes the following

$$\mathrm{Ric}(X) = (2m-1)X - 2\eta(X)\xi - AX + hSX - S^2X.$$

Since (M, g, ξ, ρ) is a Hopf–Ricci soliton and has an $\mathfrak{A}$-principal normal vector field, it satisfies

$$\begin{aligned}\rho &= \frac{1}{2}(\mathcal{L}_\xi g)(\xi, \xi) + \mathrm{Ric}(\xi, \xi) \\ &= g(\mathrm{Ric}(\xi), \xi) \\ &= 2(m-1) + h\alpha - \alpha^2,\end{aligned}$$

where we have used $A\xi = -\xi$ for $\mathfrak{A}$-principal unit normal vector field.

When the unit normal N is $\mathfrak{A}$-isotropic, the Ricci tensor becomes

$$\mathrm{Ric}(X) = (2m-1)X - 3\eta(X)\xi + g(AX, N)AN + g(AX, \xi)A\xi + hSX - S^2X.$$

Since (M, g, ξ, ρ) is a Hopf–Ricci soliton and N is $\mathfrak{A}$-isotropic, it satisfies

$$\begin{aligned}\rho &= \frac{1}{2}(\mathcal{L}_\xi g)(\xi, \xi) + \mathrm{Ric}(\xi, \xi) \\ &= g(\mathrm{Ric}(\xi), \xi) \\ &= 2(m-2) + h\alpha - \alpha^2.\end{aligned}$$

This completes the proof of our Lemma 7.1. $\square$

Now let us prove our Main Theorem 2 in the introduction. Let (M, g, ξ, ρ) be a Hopf–Ricci soliton real hypersurface in the complex quadric Q^m. Then Lemma 7.1 (i) for the $\mathfrak{A}$-principal unit normal N becomes

$$\{1 - (h\alpha - \alpha^2)\}X - 2\eta(X)\xi - AX + hSX - S^2X + \frac{1}{2}(\phi S - S\phi)X = 0. \tag{7.3}$$

On the other hand, the Hopf–Ricci soliton real hypersurface (M, g, ξ, ρ) satisfies the condition $\mathrm{Ric}\cdot\phi + \phi\cdot\mathrm{Ric} = \kappa\phi$, $\kappa = 2\rho$, then by (i) in our Main Theorem 1 for $\mathfrak{A}$-principal unit normal N such a hypersurface M is locally congruent to a tube over a totally geodesic and totally real submanifold S^m in Q^m. Then by Suh [18] and [19], we know that the tube is characterized by $S\phi + \phi S = \epsilon\phi$, $\epsilon = -\frac{2}{\alpha}$. The expression of the shape operator S of M in the complex quadric Q^m becomes

$$S = \begin{bmatrix} \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -\frac{2}{\alpha} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & -\frac{2}{\alpha} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

So first we consider the following case:

Case 1. N is $\mathfrak{A}$-principal.

Now we consider an eigen vector $X \in T_\lambda$, $\lambda = 0$. Then $X \in V(A) \oplus JV(A)$. In such a case we can divide into 3 cases $AX = X$, $AX = -X$ and $AX = \frac{1}{\sqrt{2}}Y - \frac{1}{\sqrt{2}}Z$ for some $Y \in V(A)$ and $Z \in JV(A)$. Using these properties in (7.3), we have three subcases as follows:

Subcase 1.1. $X \in V(A) \cap T_zM$, $z \in M$.

In this case $AX = X$. Since $X \in T_\lambda$, $\lambda = 0$, we know that $h\alpha - \alpha^2 = \frac{1}{\alpha}$. On the other hand, from the expressions of the tube over S^m, we know that $h - \alpha = (m-1)(-\frac{2}{\alpha})$. Then $\alpha = -\sqrt{2}\cot\sqrt{2}r = -\frac{1}{2(m-1)}$. Then the radius of the tube is given by $r = \frac{1}{\sqrt{2}}\cot^{-1}\big(\frac{1}{2\sqrt{2}(m-1)}\big)$.

Subcase 1.2. $X \in JV(A) \cap T_zM$, $z \in M$.

In this case $AX = -X$. Since $X \in T_\lambda$, $\lambda = 0$, we know that

$$\{1 - (h\alpha - \alpha^2)\}X + X + \frac{1}{\alpha}X = 0.$$

Then $\frac{1+2\alpha}{\alpha} = h\alpha - \alpha^2$. But from the expressions of the shape operator S of M in Q^m, we also have $(h-\alpha)\alpha = -2(m-1)$. From this, it follows that $\alpha = -\sqrt{2}\cot\sqrt{2}r = -\frac{1}{2m}$. Then the radius of the tube is given by $r = \frac{1}{\sqrt{2}}\cot^{-1}\big(\frac{1}{2\sqrt{2}m}\big)$.

Subcase 1.3. $X = \frac{1}{\sqrt{2}}Y + \frac{1}{\sqrt{2}}Z$ for $Y \in V(A) \cap T_zM$, $Z \in JV(A) \cap T_zM$, $z \in M$.

In this case $AX = \frac{1}{\sqrt{2}}Y - \frac{1}{\sqrt{2}}Z$. Then it follows that

$$\{1 - (h\alpha - \alpha^2)\}(\frac{1}{\sqrt{2}}Y + \frac{1}{\sqrt{2}}Z) - (\frac{1}{\sqrt{2}}Y - \frac{1}{\sqrt{2}}Z) + \frac{1}{\alpha}(\frac{1}{\sqrt{2}}Y - \frac{1}{\sqrt{2}}Z) = 0.$$

From this, comparing the coefficients of the vector fields Y and Z respectively, we have the following

$$h\alpha - \alpha^2 = \frac{1}{\alpha}$$

and

$$h\alpha - \alpha^2 = 2 - \frac{1}{\alpha}.$$

Then we have $\alpha = -\sqrt{2}\cot\sqrt{2}r = 1$, which gives a contradiction. So this case can not happen.

Next we consider a Ricci soliton Hopf real hypersurface (M, g, ξ, ρ) in the complex quadric Q^m with $\mathfrak{A}$-isotropic unit normal N as follows:

Case 2. N is $\mathfrak{A}$-isotropic.

In this case we assume that the Ricci soliton (M, g, ξ, ρ) is non-minimal. It is known that the Hopf–Ricci soliton (M, g, ξ, ρ) satisfies the pseudo-anti commuting Ricci tensor property. Then by (ii) in our Main Theorem 1 for $\mathfrak{A}$-isotropic unit normal N, M is locally congruent to a tube of radius r over $\mathbb{C}P^k$ in Q^{2k}. So the shape operator S of the pseudo-anti commuting Hopf hypersurface in Q^m can be expressed as

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cot r & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cot r & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\tan r & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -\tan r \end{bmatrix}.$$

On the other hand, as N is $\mathfrak{A}$-isotropic we know that

$$\frac{1}{2}(\mathcal{L}_\xi g)(X, Y) + \mathrm{Ric}(X, Y) = \rho g(X, Y)$$

where the Ricci soliton constant ρ is given by

$$\rho = 2(m-2) + h\alpha - \alpha^2$$

in Lemma 7.1. Then we get

$$\begin{aligned} &\frac{1}{2}(S\phi - \phi S)X + (2m-1)X - 3\eta(X)\xi + g(AX, N)AN + g(A\xi, X)A\xi + hSX - S^2X \\ &= \{2(m-2) + h\alpha - \alpha^2\}X. \end{aligned}$$

From this, putting $SX = \cot rX$, $S\phi X = \cot r\phi X$, for $X \perp \mathrm{Span}\{\xi, A\xi, AN\}$, we have

$$3 + h\cot r - \cot^2 r = h(\cot r - \tan r) - (\cot r - \tan r)^2.$$

This becomes

$$\tan^2 r + h\tan r + 1 = 0,$$

where the trace h is given by $h = \alpha + 2(k-1)\{\cot r - \tan r\} = (2k-1)(\cot r - \tan r)$. So it follows that $h\tan r = (2k-1)(\cot r - \tan r)\tan r = 2k - 1 - (2k-1)\tan^2 r$. Then $\tan^2 r = \frac{k}{k-1}$, that is, $r = \tan^{-1}\sqrt{\frac{k}{k-1}}$, where $2(k-1)$ denotes the multiplicities of the principal curvatures $\cot r$ and $-\tan r$ respectively. Of course, this kind of tube becomes non-minimal and pseudo-Einstein as in the following remark:

Remark 7.2. We check whether the Ricci tensor of the tube M over a totally geodesic $\mathbb{C}P^k$ in Q^m, $m = 2k$, mentioned in Suh [18] and [19] satisfies the notion of pseudo-Einstein or not. By a theorem due to Suh [18] and [19], the shape operator S commutes with the structure tensor ϕ, that is, $S\phi = \phi S$. In this case we know that the normal vector field N of M in Q^{2k} is $\mathfrak{A}$-isotropic. Then $g(AN, N) = 0$, $g(A\xi, \xi) = 0$, $g(A\xi, N) = 0$. So let us suppose that M is pseudo-Einstein. Then for any vector field X on M the Ricci tensor Ric becomes the following

$$\begin{aligned}\mathrm{Ric(X)} &= (2m-1)X - 3\eta(X)\xi + g(AX,N)AN + g(AX,\xi)A\xi + hSX - S^2X \\ &= aX + b\eta(X)\xi\end{aligned} \tag{7.4}$$

for some constant real numbers $a, b \in R$. Putting $X = \xi$ into (7.4), we have

$$(a+b)\xi = (2m-4)\xi + (h\alpha - \alpha^2)\xi,$$

where

$$\begin{aligned}h\alpha - \alpha^2 &= \{2\cot 2r + 2(k-1)(\cot r - \tan r)\}2\cot 2r - (2\cot 2r)^2 \\ &= 2(k-1)(2\cot 2r)^2 = 8(k-1)\cot^2 2r.\end{aligned}$$

From this, together with $m = 2k$, we have

$$a + b = 4(k-1)\{1 + 2\cot^2 2r\}. \tag{7.5}$$

For any X orthogonal to the vector fields ξ, $A\xi$, and AN such that $SX = \cot rX$ the equation (7.4) becomes

$$aX = (4k-1)X + \{h\cot r - \cot^2 r\}X,$$

where

$$\begin{aligned}h\cot r - \cot^2 r &= \{2\cot 2r + 2(k-1)2\cot 2r\}\cot r - \cot^2 r \\ &= (2k-1)(\cot r - \tan r)\cot r - \cot^2 r \\ &= 2(k-1)\cot^2 r - (2k-1).\end{aligned}$$

From this, together with (7.5), we have

$$\begin{aligned}a &= 2k + 2(k-1)\cot^2 r, \\ b &= -2k + 2(k-1)\tan^2 r.\end{aligned}$$

Putting $X = A\xi$ into (7.4), and using the properties $g(A\xi, \xi) = 0$, $A^2\xi = \xi$ and $SA\xi = 0$, we have

$$aA\xi = (2m-1)A\xi + A\xi = 2mA\xi = 4kA\xi.$$

From this, together with (7.5), it follows that $a = 4k$ and $b = -4 + 8(k-1)\cot^2 2r$. Comparing with the previous values of a and b, we conclude that

$$\cot^2 r = \frac{k}{k-1}.$$

Summing up our discussions, we conclude that the tube of radius $r = \cot^{-1}\sqrt{\frac{k}{k-1}}$ around $\mathbb{C}P^k$ in Q^{2k} is a pseudo-Einstein Hopf real hypersurface in the complex quadric Q^{2k} with $\mathfrak{A}$-isotropic unit normal vector field. Of course, the constants a and b are respectively given by $a = 4k$ and $b = -4 + \frac{2}{k}$. They have been calculated as follows:

$$\begin{aligned}a &= 2k + 2(k-1)\cot^2 r \\ &= 2k + 2(k-1)\cdot\frac{k}{k-1} \\ &= 4k,\end{aligned}$$

and

$$\begin{aligned} b &= -2k + 2(k-1)\tan^2 r \\ &= -2k + \frac{2(k-1)^2}{k} \\ &= -2k + \frac{2(k^2-2k+1)}{k} \\ &= -4 + \frac{2}{k} \end{aligned}$$

respectively.

Moreover, it becomes a Ricci soliton (M, ξ, g, ρ) with Ricci soliton constant $\rho = 2(m-2) + h\alpha - \alpha^2$ in Lemma 7.1 and satisfies the condition of pseudo-anti commuting Ricci soliton, that is, $\mathrm{Ric}\cdot\phi + \phi\cdot\mathrm{Ric} = \kappa\phi$, $\kappa = 2\rho$. Of course, the trace h is non-vanishing, that is, M is non-minimal.

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